作业1

Problem 1　m, n.　A　$m\times n$ 矩阵，有左逆无左逆.

rank = m

rank = n

当 $m<n$ 时　$AB = I_m$

$CA = I_n$

举例子.　$A\ B$

$v_{m+1}\ v_{n+L} \cdots v_{n+n-m}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 1 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 1 & \cdots & \vdots & * & * & * \\ 0 & 0 & 0 & \ddots & 0 & & * & \\ \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & * & * & * \end{pmatrix}_m$$

n

当 $m=n$ 时　? 无.

左逆 = 右逆

当 $m>n$ 时　无.

行最简形

主元

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ & 0 & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \end{pmatrix}$$

rank ≤ min{m, n}

$$\begin{pmatrix} \lambda_1 \cdots \lambda_m \\ \\ \end{pmatrix}$$

$CA = (0\ 0\ 0 \cdots 1)\ 0$

将 A 右乘 B　　AB 表示对 A 的列作变换 $\rightarrow \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ n个元素

列是向量

左乘 C　　CA　　A 的变换 $\rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ n个元素

$\boxed{\text{rank}}$

(2). B_1 为 A 右逆，B_2 为 A 左逆 $\Rightarrow B_1 = B_2$.

$$\begin{array}{c} B_2 A B_1 = (B_2 A) B_1 = B_1 \\ \| \\ B_2(AB_1) = B_2 \end{array}$$

若 B 是 A 右逆 $\Rightarrow$ B 是 A 左逆（方阵）

(4). $n\times n$ 矩阵 $AB = I \Rightarrow BA = I$.

$AB = I$

$AB = I \Rightarrow \operatorname{rank} B = n$.

约

$\Rightarrow B$ 有右逆 $\Rightarrow C$

$$BAB = B(AB) = B$$

$$\Rightarrow BAB - \underbrace{B}_{IB} = 0 \Rightarrow (BA - I)B = 0.$$

$$(BA-I)\underbrace{BC}_{I} = 0\cdot C = 0, \Rightarrow BA - I = 0.$$

n×n 的矩阵 A，是否有可能两个左逆？ 2 min

$B_1 A = I$ $\quad AB_1 = I$ $\quad B_1 A B_2 = B_2$

$B_2 A = I$ $\quad AB_2 = I$ $\quad B_1 = B_1$

问题 2. 4个基本子空间

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

rank $A = 2 = \dim(\mathrm{Row}(A))$ $\quad \dim(\mathrm{Null}(A)) = 0$

$\dim(\mathrm{Null}(A^T)) = 1$

$$\begin{array}{ccccc} x_1 & x_2 & & & \\ \begin{pmatrix} 1 & 4 & 7 & 3 & 0 \\ 2 & 5 & 8 & 9 & 2 \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

Step 1. 先确定 rank A.

Step 2. 找出 rank A 个列线性无关

Step 3. 对未选中的列，作 0/1

3个零空间的元. $\begin{pmatrix} x_1 \\ x_2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \vdots \\ 0 \\ 3 \\ \end{pmatrix} \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 2 \end{pmatrix}$

$\mathbb{R}^t$

2, $x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}7\\8\end{pmatrix} = 0$

$$\begin{pmatrix} 1 & 4 & 7 & 3 & 0 \\ 2 & 5 & 8 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$x_3 \begin{pmatrix} * \\ * \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} * \\ * \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

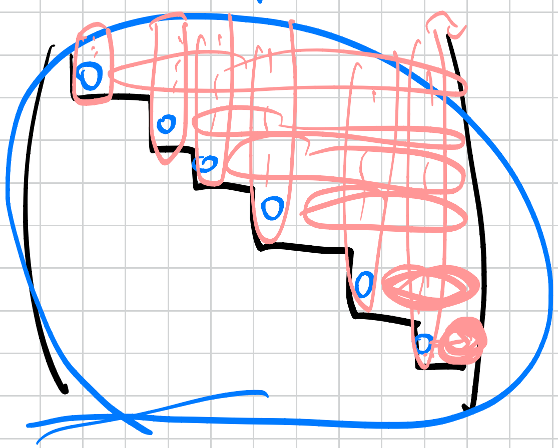
dim N = t - rank

dim Null + dim Row A = n

(2) $\boxed{\text{rank } A} = \boxed{\text{dim Row}(A)}$

Cw

Rw



$\text{rank} A = \dim \text{Row} A.$ $\quad \text{rank} A' = \dim \text{Row} A'$ $\quad \text{rank} A'' = \dim \text{Row} A''$

$A \xrightarrow{\text{行}} \begin{pmatrix} \text{阶梯形} \end{pmatrix} \xrightarrow{\text{列}} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

(3) 空间正交.

$A_{m\times n}$

$\text{Col}(A) \subseteq \mathbb{R}^m \quad \longleftrightarrow \quad \text{Row}(A) \subseteq \mathbb{R}^n$

$\text{Null}(A) \subseteq \mathbb{R}^n \quad \text{Null}(A^T) \subseteq \mathbb{R}^m$

$A, B \subseteq \mathbb{R}^m$

$\left(\forall x \in A \quad \forall y \in B \quad x^T y = 0 \right)$

则 $A \perp B$

$\perp$：正交.

$n \begin{pmatrix} \; \end{pmatrix} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

n

$\mathrm{Null}\,A = \{x \mid Ax = 0\}$

$\forall\, y \in \mathrm{Row}\,A \Rightarrow y^T x = 0$

$x \in \mathrm{Null}\,A.$

$$A = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{pmatrix} \qquad Ax = \begin{pmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_m^T x \end{pmatrix} = 0$$

$\mathbb{R}^n$: 维度为 m

$\boxed{\text{互补}}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

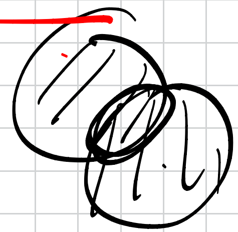
$A \perp B$

$a \qquad b \qquad a+b=n$

$x^T x = 0 \Rightarrow x = 0.$

不仅有 $A \perp B$ 且有 $\dim A + \dim B = m$ 　直和为全空间

$A \cap B \neq 0.$



$\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$

$\Rightarrow \dim(A+B) = \dim(A) + \dim(B) = m$

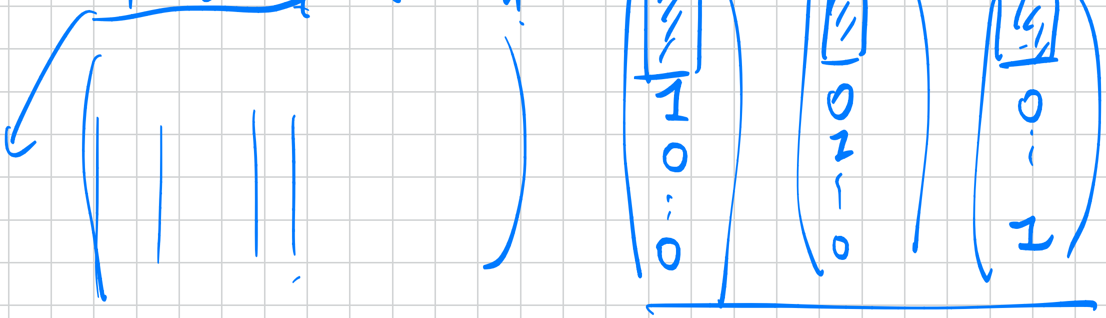
3) 证:

$$\dim(\underbrace{\mathrm{Null}(A)}_{\mathbb{R}^n}) + \dim(\underbrace{\mathrm{Row}A}_{\mathbb{R}^n}) = \underline{n} \quad \checkmark$$

$$\dim(\underbrace{\mathrm{Null}\,A^T}_{\mathbb{R}^m}) + \underbrace{\mathrm{Rank}\,A}_{\mathbb{R}^m} = \underline{m}$$

A 确定 $\dim \mathrm{Row}A = \mathrm{Rank}A$ 个列无关

$v_1\ v_2 \cdots v_t \quad v_{t+1} \sim v_n$



$v_1 \sim v_t$ basis

$$\dim \mathrm{Null}(A) = n - t = n - \mathrm{rank}A = n - \dim(\mathrm{Row}A)$$

(4). 证: rank $A \geq$ rank (AB)

$$\dim \mathrm{Col}(A) \qquad \dim \mathrm{Col}(AB)$$

$$\mathrm{Col}(AB) \subseteq \mathrm{Col}(A) \quad \checkmark$$

(5). $\text{rank } A + \dim \mathrm{Null}\, A = n.$ $\qquad A_{m\times n}$

$$\text{rank } A + \dim \mathrm{Null}\, A^T = m$$

$$\text{rank } B \leq \overbrace{\boxed{n - \text{rank } A}}^{\dim \mathrm{Null}\, A} + \text{rank}\, AB$$

$$\underbrace{\text{rank } B - \text{rank } AB}_{\dim \mathrm{Null}\, BA} \leq \underbrace{n - \text{rank } A}_{\dim \mathrm{Null}\, A}$$

$$\mathrm{Null}_B A = \{ x \in \mathrm{Row}\, B \mid Ax = 0 \}$$

$$\mathrm{rank}\, I_n - \mathrm{rank}\, A I_n$$

$$\|$$

$$\dim \mathrm{Null}\, A$$

可写成某子空间 dimension

$$\mathrm{Null}\, A = \{ x \in \mathrm{Row}\, I_n \mid Ax = 0 \}$$

I 的行空间 $\mathrm{Row}\, I_n = \mathbb{R}^n$

B的行空间 $\quad \mathrm{Row}\, B$

$$\supseteq$$

AB 的行空间 是 Row B 的一部分

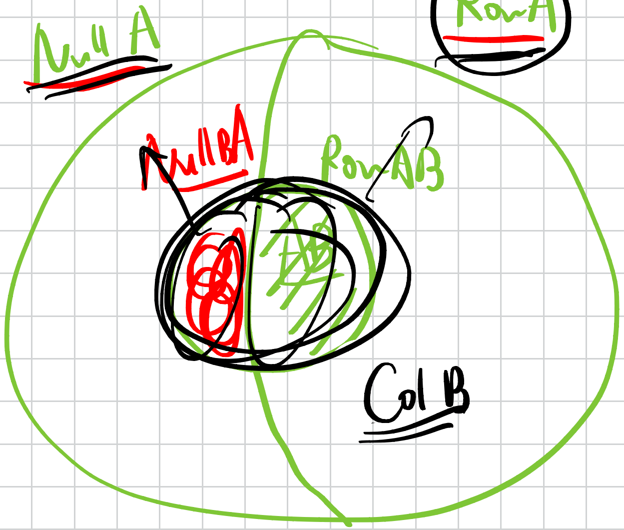
AI 的行空间 $\mathrm{Row}\, AI = \mathrm{Row}\, A$. $\mathbb{R}^n$ 的一部分

$m \times n$

n维

Row A

Col B → n维



$\mathbb{R}^n$

$$\boxed{\mathrm{Row}\, B = \mathrm{Row}\, AB + \mathrm{Null}_B A}$$

$\mathrm{Col}\,B \subseteq \mathbb{R}^n$ rank AB $B: n \times l$

$\mathrm{Row}\,A \cap \mathrm{Col}\,B \subseteq \mathrm{Row}\,A \subseteq \mathbb{R}^n$

$\mathbb{R}^n = \mathrm{Row}\,A \oplus \mathrm{Null}\,A$

$\mathrm{Col}\,B = (\mathrm{Row}\,A \cap \mathrm{Col}\,B) \oplus (\mathrm{Null}_B A)$?
rank AB ?

$\mathrm{Null}_B A = \{x \mid Ax = 0 \quad x \in \mathrm{Col}(B)\} \subseteq \mathrm{Null}\,A$ rank B

1. $\oplus$

$Ax = 0$ $\quad x \in \mathrm{Row}\,A \cap \mathrm{Col}\,B$

$\Rightarrow x = 0$

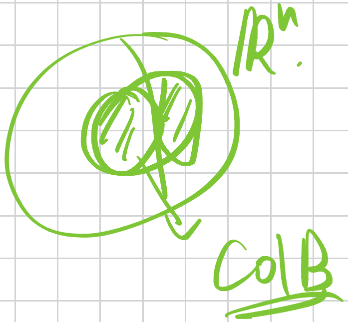
$x \in \mathrm{Null}\,A$

$v \in \mathrm{Col}\,B \subseteq \mathbb{R}^n = \mathrm{Row}\,A \oplus \mathrm{Null}\,A$

$\mathrm{Null}_B A = \mathrm{Col}\,B \cap \mathrm{Null}\,A$

$\dim \mathrm{Null}_B A + \dim(\mathrm{Row}\,A \cap \mathrm{Col}\,B) \leq \dim \mathrm{Col}\,B$

$\dim(\mathrm{Col}\,B \cap \mathrm{Null}\,A)$ $\quad \mathbb{R}^n$



$V = A \oplus B \qquad C \subseteq V$

$$\boxed{\dim C = \dim C \cap A + \dim C \cap B}$$

$V = \mathbb{R}^2$.

$A = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\}$

$B = \left\{ \begin{pmatrix}0\\1\end{pmatrix} \right\}$.

$C = \left\{ \begin{pmatrix}1\\1\end{pmatrix} \right\}$

定理: A B, f

空间 空间函数

$f: A \to B$ 满足: $f(v_1+v_2)=f(v_1)+f(v_2)$

$f(cv)=cf(v)$, $c\in\mathbb{R}$

$\mathbb{R}$

Row A　　Null(A)

则 $\dim \operatorname{Im} f + \dim \operatorname{Ker} f = \dim A$ (同态定理)

在此令 $A=\mathbb{R}^n$

$\operatorname{Ker} f = \operatorname{Null} A$

$f: v \mapsto Av$

则 $\operatorname{Im} f = \operatorname{Col} A$.

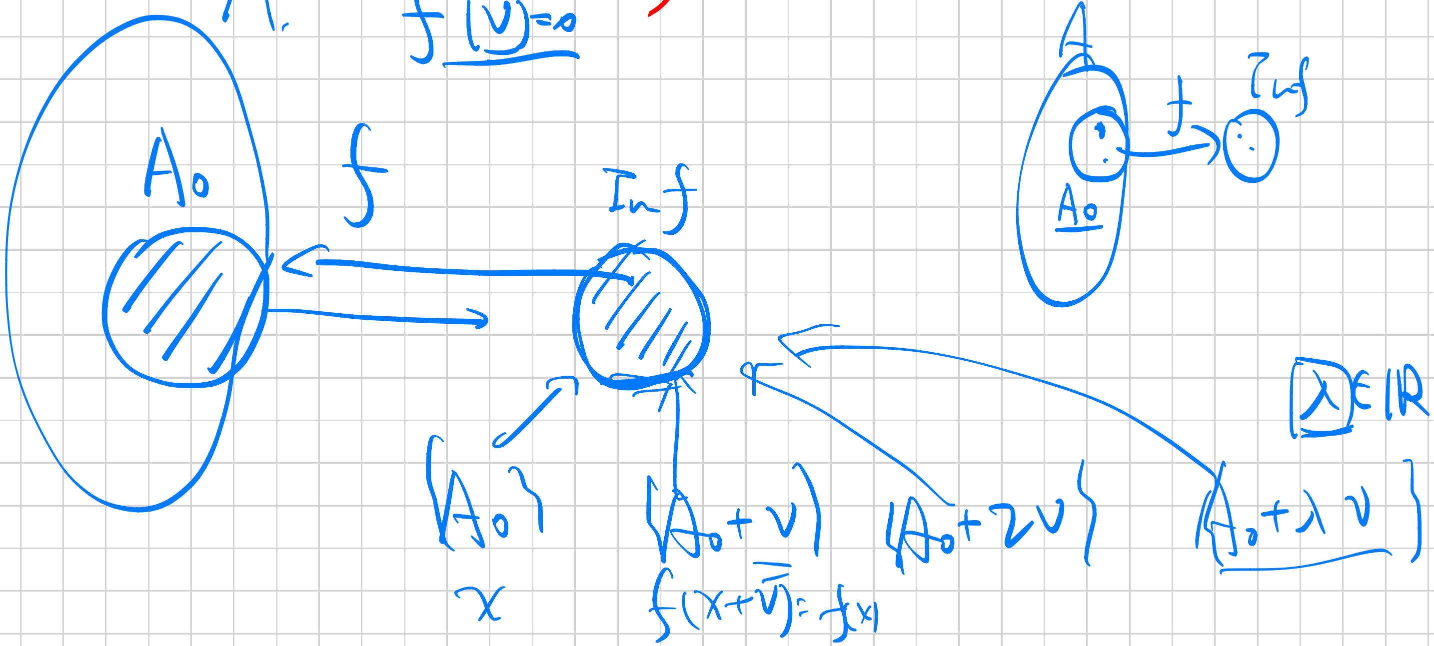
$\operatorname{Im} f = \{f(x)\}$ 像集

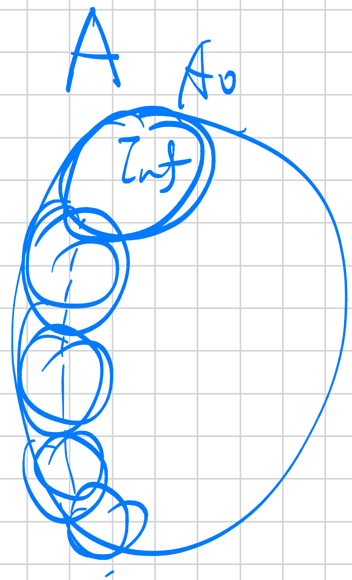
$\dim \operatorname{Col} A + \dim \operatorname{Null} A = n$

$\operatorname{rank} A = \dim \operatorname{Row} A$

$\operatorname{Ker} f = \{x\in A \mid f(x)=0\}$ 核

$f(v)=0$





$v_i \in \ker f$ $\quad$ n维

$$A = A_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \cdots v_m$$

$\mathrm{Im} f$

$$\dim A = \dim A_0 + \dim \ker f$$

$A \xrightarrow{f} B$ $\quad$ 取出 A 的 basis : $v_1 \cdots v_t$. $\quad$ 设 B 的 basis 是 $u_1 \sim u_m$

$$f(v_i) = (u_1 \cdots u_m)\begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{pmatrix} \qquad (f(v_1) \cdots f(v_t)) \Leftrightarrow (\ , \ \cdots \ ) \ (u_1 \cdots u_m)$$

$\dim A = t$

$C: m \times t$ 矩阵. $\qquad C$ 最大.

$\dim \mathrm{Im} f = \mathrm{Rank}\, C$.

$\mathrm{Rank}\, C + \underset{\dim}{\mathrm{Null}}\, C = t$

$\dim \ker f \,?\, \mathrm{Null}\, C$.

$$0 = f(c_1 v_1 + \cdots + c_t v_t)$$
$$= f(v_1)\cdot c_1 + f(v_2)\cdot c_2 + \cdots + f(v_t)\cdot c_t$$
$$= (f(v_1), f(v_2) \cdots f(v_t))\begin{pmatrix}c_1 \\ \vdots \\ c_t\end{pmatrix}$$

$$= (u_1 \cdots u_m)\left(C \begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} \right)$$

$$\| $$

$$0$$

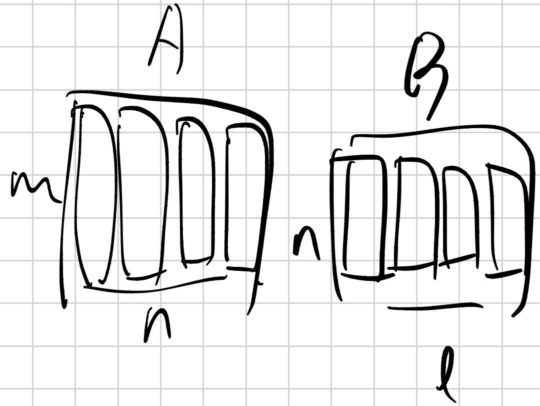
$A : m \times n$

$B : n \times l$

$\operatorname{rank} B - \operatorname{rank} AB \leq n - \operatorname{rank} A = \dim \operatorname{Null} A$

$\ker f \subseteq \operatorname{Null} A$

$$f : \operatorname{Col} B \to \mathbb{R}^m$$
$$v \mapsto Av$$



$\operatorname{Null} A : \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

$\cup|$

$\ker f : \{ x \in \operatorname{Col} B \mid Ax = 0 \}$

$\dim \operatorname{Im} f + \dim \ker f = \dim \operatorname{Col} B = \operatorname{rank} B$

$//$

$\operatorname{Col} B : B \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \mapsto Av \quad AB \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \quad \operatorname{Im} f = \operatorname{Col} AB$

$\operatorname{rank} AB$

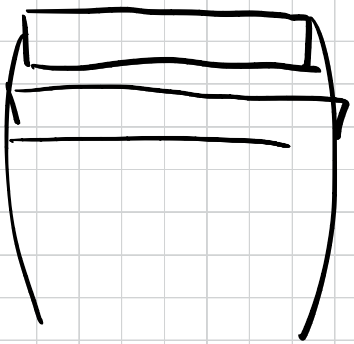
$\operatorname{rank} B - \operatorname{rank} AB = \dim \ker f$

$\leq n - \operatorname{rank} A$

$\square$

(1) 算 $GL_n(p)$ 和 $SL_n(p)$

$$\left|\{\, n\times n \text{ 可逆矩阵} \mid \text{元素模 } p \text{ 的整数} \,\}\right|$$



r_1 不能为 0　　$r_1 : p^n-1$

r_2 不能 $\in \mathrm{span}\{r_1\}$　　$r_2 : p^n-p$

r_3 不能 $\in \mathrm{span}\{r_1, r_2\}$　　$r_3 : p^n-p^2$

$\vdots$

$r_n : p^n-p^{n-1}$

$$|GL_n(p)| = (p^n-1)\cdots(p^n-p^{n-1})$$

$$SL_n(p) = \{A\in GL_n(p) \mid \det A = 1\} \Rightarrow |SL_n(p)| = \frac{|GL_n(p)|}{p-1}$$

$GL_n(p) \ni \underset{a}{A_1}$　　A_1 第 1 行 $\times 2 \Rightarrow A_2$　　$\det = 2a$

$\times 3 \Rightarrow A_3$　　$\det = 3a$

$\vdots$

$\times(p-1)$ A_{p-1} $\det = (p-1)a$

$\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix} = 2\begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix}$

$GL_n(p)$ 分轨道

A

1 $\{A_1, A_2, \cdots, A_{p-1}\}$

2 $\{B_2 \}$

$\vdots$

$p-1$ $\{ \}$

$A \in \{A_1, \cdots, A_{p-1}\}$

$\in \{B_1, \cdots, B_{p-1}\}$

$A_i =$ A 第一行 $\times i$

$B_k = A$ $\quad B_{k\times s} =$ A 第一行 $\times \frac{s}{k}$

2 $a, b \in G$ $aba^{-1} = b^r$

$a^i b a^{-i} = b^{r^i}$

$a \, a^{i-1} b a^{-(i-1)} a^{-1} = a b^{r^{i-1}} a^{-1} = \underline{a b^{r^{i-1}}} a^{-1} = b^{r^i}$

$\underbrace{aba^{-1}aba^{-1} \cdots aba^{-1}}_{\text{重复 } r^{i-1} \text{ 次}} = \underbrace{b^r \cdot b^r \cdots b^r}_{r^{i-1}} = \boxed{\underline{b^{r^i}}}$ ✓

3. $(ab)^2 = a^2 b^2$ 证明 G 交换群.

$\cancel{a}bab = a^2b^2 = \cancel{a}abb \Rightarrow \underline{ba = ab}$ ✓

$\exp(G)=2.$ $\forall g\in G$ $g^2=1$.

$(ab)^2=a^2b^2=1$ $\boxed{交换}$.

4\. $|G|=$偶 $\exists a\neq e.$ $a^2=e$.

$a^2\neq e$ $x^{-1}\neq x$.

e. $\{x_i\leftrightarrow x_j\}$ $\underline{x_i x_j=1}$.

5\. $\underline{|a|=4}$ $\underline{a^4=1}$ $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^4=1$, $|b|=3$ $b^3=1$.

$|ab|=\infty$ $ab=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.