Homework 1 作业 1

Each problem has 20 pts as a full mark. 每个问题按照20分为满分。

Problem 1 问题 1

(1) For fixed $m, n \in \mathbf{N}_+$, give an example: $A \in \mathbf{R}_{m\times n}$, and A doesn't have a left inverse while has a right inverse. 对于固定的正整数m, n，给出一个$m \times n$的实数矩阵A，使得其有右逆矩阵但是没有左逆矩阵

(2)Prove: if $AB_1 = I, B_2A = I$, then $B_1 = B_2$. 证明：若B_1为A右逆，B_2为A左逆，则$B_1 = B_2$

(3) For fixed $m, n \in \mathbf{N}_+$, discuss whether there exists $A \in \mathbf{R}_{m\times n}$ such that it has both left inverse and right inverse. 对于固定的正整数m, n，讨论是否存在$m \times n$的矩阵A，同时有左逆和右逆。

(4) Prove: for $A, B \in \mathbf{R}_{n\times n}$, if $AB = I$, then $BA = I$. 证明：对于$n \times n$的矩阵A, B，若$AB = I$，则$BA = I$

Problem 2 问题 2

Definition of $rank(A)$ is the dimension of the column space of A. $rank(A)$的定义是A的列空间的维度。

(1) For

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

Calculate: $rank(A), dim(Null(A)), dim(Row(A)), dim(Null(A^T))$.

计算：$rank(A), dim(Null(A)), dim(Row(A)), dim(Null(A^T))$

(2) Prove: $rank(A) = dim(Row(A))$. 证明：$rank(A) = dim(Row(A))$

(Hint: first prove: Gauss Elimilation doesn't change $rank(A)$ and $dim(Row(A))$.Then do the RREF situation. 提示：先去证明：高斯消元不会改变$rank(A)$和$dim(Row(A))$， 然后考虑行最简形的情况)

(3) Prove: for $A \in \mathbf{R}_{m\times n}$, $rank(A) + dim(Null(A^T)) = m, dim(Row(A)) + dim(Null(A)) = n$. 证明：对$m \times n$的实数矩阵$A$，$rank(A) + dim(Null(A^T)) = m, dim(Row(A)) + dim(Null(A)) = n$

(4) Prove: for $A \in \mathbf{R}_{m\times n}, B \in \mathbf{R}_{n\times l}, rank(A) \geq rank(AB)$. 证明：对$m \times n$的实数矩阵$A$，$n \times l$的实数矩阵$B$，证明：$rank(A) \geq rank(AB)$

(5) Prove: for $A \in \mathbf{R}_{m\times n}, B \in \mathbf{R}_{n\times l}, rank(A) + rank(B) \leq n + rank(AB)$. 证明：对$m \times n$的实数矩阵$A$，$n \times l$的实数矩阵$B$，证明：$rank(A) + rank(B) \leq n + rank(AB)$

Problem 3 问题 3

(1) Calculate the order of following groups: 计算下列群的阶：

(a) $GL_n(p)$; (b) $SL_n(p)$.

(2) Let G be a group, $a, b \in G$. If $aba^{-1} = b^r$, prove that

$$a^i b a^{-i} = b^{r^i}.$$

设 G 为群，$a, b \in G$。若 $aba^{-1} = b^r$，证明

$$a^i b a^{-i} = b^{r^i}.$$

(3) Let G be a group. If $\forall a, b \in G$, $(ab)^2 = a^2b^2$. Prove that G is abelian.
Show that if $\exp(G) = 2$, then G is abelian. ($\exp(G)$ is the smallest integer n such that $\forall g \in G$, $g^n = e$.)

设 G 为群。若对任意 $a, b \in G$，有 $(ab)^2 = a^2b^2$，证明 G 为交换群。
并证明若 $\exp(G) = 2$，则 G 为交换群。 （$\exp(G)$ 为最小正整数 n，使得对任意 $g \in G$，$g^n = e$。）

(4) Assume $|G|$ is even. Prove that there exists $a \neq e$ in G such that

$$a^2 = e.$$

设 $|G|$ 为偶数，证明存在 $a \neq e$ 使得

$$a^2 = e.$$

(5) Consider $a, b \in SL_2(\mathbb{Q})$, where

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Prove that: (a) $|a| = 4$; (b) $|b| = 3$; (c) $|ab| = \infty$.

设 $a, b \in SL_2(\mathbb{Q})$，其中

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

证明： (a) $|a| = 4$； (b) $|b| = 3$； (c) $|ab| = \infty$。