# Naive Set Theory

## cardinal arithmetic and number

**Def 1** We use $card\ A$ to describe the comparative sizes of a set $A$, which is called the cardinal number of $A$.

**Def 2** we use $=, <, >, \leq, \geq$ to descirbe the order of cardinal number, which defined by following sentences.

$$\begin{aligned} card\ A = card\ B &\iff A \sim B \\ card\ A > card\ B &\iff A \succ B \\ card\ A < card\ B &\iff A \prec B \\ card\ A \geq card\ B &\iff A \succeq B \\ card\ A \leq card\ B &\iff A \preceq B \end{aligned}$$

**Def 3** $A, B$ ars disjoint sets and $card\ A = a, card\ B = b$, then we use $a + b$ to describe $card\ A \cup B$

*Remark*: If we use $C \sim A, D \sim B$, and $C, D$ are pairwise disjoint, then $card\ C \cup D = a + b$, which means $a + b$ is well-defined and it's independent of the choice of $A, B$

**Prop 1**

- commutative: $a + b = b + a$
- associative: $a + (b + c) = (a + b) + c$

  proof: use the definition of set union

**Exe 1** $a, b, c, d$ are cardinal numbers of some set. If $a \leq b, c \leq d$, then $a + c \leq b + d$

proof: assume $card\ A = a, card\ B = b, card\ C = c, card\ D = d$. $A, B, C, D$ are all disjoint, then

$$\begin{aligned} \exists B_1 \subset B, A \sim B_1 \\ \exists D_1 \subset D, C \sim D_1 \end{aligned}$$

for $B_1, D_1$ are disjoint,we have $a + c = card\ A \cup C = card\ B_1 \cup D_1$, and we have $b + d = card\ B \cup D$. because $B_1 \cup D_1 \preceq B \cup D$, we have $a + c \leq b + d$. $\square$

**Def 4** for $\{A_i\}$ is a correspondingly indexed family of pairwise disjoint sets such that $card\ A_i = a_i$, then

$$\sum_i a_i = card\ \cup_i A_i$$

**Def 5** $A, B$ ars sets and $card\ A = a, card\ B = b$, then we use $ab$ to describe $card\ A \times B$

**Prop 2**

- commutative: $ab = ba$
- associative: $a(bc) = (ab)c$
- multiplication distribute over addition $a(b + c) = ab + ac$

  proof: use the definition of set union and Cartesian product

**Exe 2** $a, b, c, d$ are cardinal numbers of some set. If $a \leq b, c \leq d$, then $ac \leq bd$

proof: similar to *Exe1*

**Def 6** for $\{A_i\}$ is a correspondingly indexed family of sets such that $card\ A_i = a_i$, then

$$\prod_i a_i = card\ \times_i A_i$$

**Exe 3** if $\{a_i\}, \{b_i\}, i \in I$ are families of cardinal numbers such that $a_i < b_i$ for each $i \in I$ ,then $\sum_i a_i < \prod_i b_i$

proof: assume that $\sum_i a_i \geq \prod_i b_i$, then for pairwize disjoint sets $A_i, B_i, card\ A_i = a_i, card\ B_i$, there exits an onto map:

$$f : \cup_i A_i \to \times_i B_i$$

for $u \in \times_i B_i$, denote $\pi_i(u)$ as the $i_{th}$ component of u

then we have $\pi_i(f(A_i)) \subset B_i$ and by $a_i < b_i$ ,there exits $v_i \in B_i - \pi_i(f(A_i))$

then $\times_i\{v_i\}$ is not in $\cup_i f(A_i)$, it's contractive.$\square$

**Def 4** for $card\ A = a, card\ B = b, a^b = card\ (A^B)$, by $A^B = \{f : f \text{ is a map from B to A}\}$

**Prop 3**

- $a^{b+c} = a^b a^c$
- $(ab)^c = a^c b^c$
- $a^{bc} = (a^b)^c$

*hint:* we can divide $f$ into two parts.

**Exe 4**

- if $a, b, c$ are cardinal numbers such that $a \leq b$, then $a^c \leq b^c$
- if $a, b$ are finite, greater than 1, and $c$ is infinite, then $a^c = b^c$

proof: we refer a result $cc = c$ then $b^c \leq c^c \leq (2^c)^c = 2^{cc} = 2^c \leq a^c$

by **Schroder Bernstein Thm.** $a^c = b^c$ $\square$

*remark:* $2^c = c^c$

**Prop 4**

- $a$ is finite and $b$ is infinte, then $a + b = b$
- $a$ is infinite, then $a + a = a$
- $a, b$ are cardinal number at least one of which is infinite, $c$ is the larger one, then $a + b = c$
- $a$ is infinite, then $aa = a$

**Exe 5**

- if $a, b$ are at least one of which is infinite, then $a + b = ab$
- if $a$ is infinite and $b$ is finite, then $a^b = a$

**Prop 5** for each set $X$, the ordinal numbers equivalent to $X$ constitute a set

**Def 5** *card* $X$ is an ordinal number $\alpha$ such that if $\beta$ is an ordinal number equivalent to $\alpha$, then $\alpha \leq \beta$

**Thm 1** (Cantor's paradox) there is not an upper bound over all ordinal number

**Exe 6** each infinite cardinal number is a limit number

**Exe 7**

- if $card\ A = a$, what is the cardinal number of the set of all one-to-one mappings of $A$ onto itself
- what is the cardinal number if the set of all countably infinite subsets of $A$

*remark*:

- continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$
- generalized continuum hypothesis: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$, for all ordinal number $\alpha$