Incidence Geometry with Polarity and Tiled Surfaces

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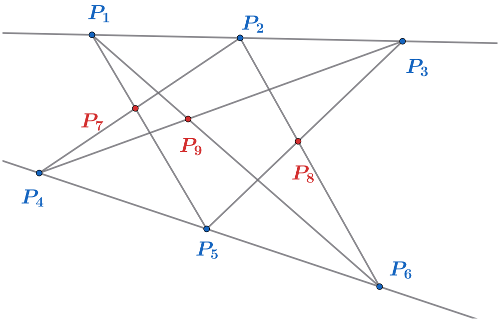
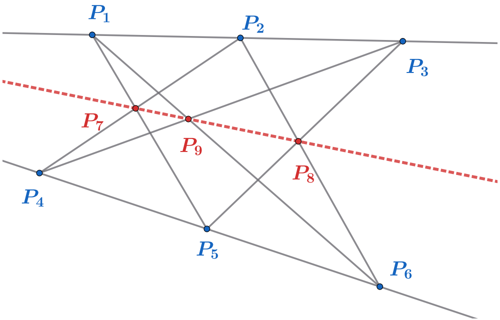
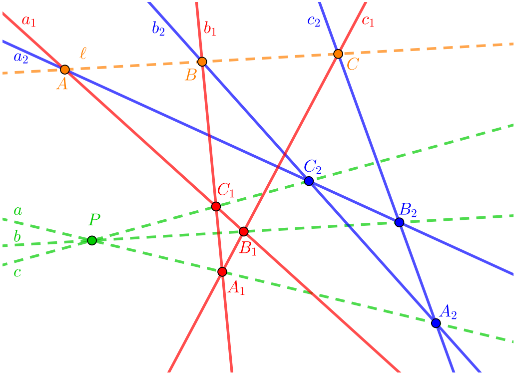
Incidence Theorem: Pappus's Theorem



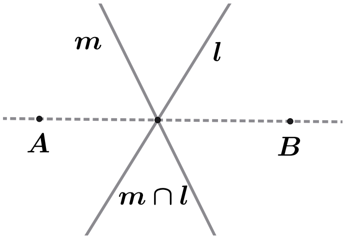
$\Longrightarrow$





a_1
a_2
ℓ
b_2
b_1
c_2
c_1
A
B
C
C_2
C_1
B_2
a
P
b
B_1
c
A_1
A_2

Coherent Tile [Fomin–Pylyavskyy, 2023]



$A,\ B,\ m \cap \ell$: collinear

$\Updownarrow$

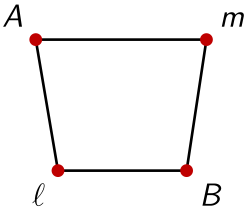
$m,\ \ell,\ (AB)$: concurrent

$\Updownarrow$

$$\frac{< m, A >< \ell, B >}{< m, B >< \ell, A >} = 1$$

This relation can be encoded by a quadrilateral containing the four elements $A,\ B,\ m,\ \ell$ and Mixed Cross Ratio:

$$\chi(A, B; m, \ell) = \frac{< m, A >< \ell, B >}{< m, B >< \ell, A >}$$



coherent tile

$\Updownarrow$

$\chi(A, B; m, \ell) = 1$

Master Theorem [Fomin–Pylyavskyy, 2023]

Consider a quadrilateral tiling T of a **closed oriented surface**, whose vertices are colored black and white in bipartite fashion.

black vertex

$\bullet$

$A \in \mathbb{P}$

associated with a **projective point**

For every edge

$$\overset{A}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\overset{h}{\circ}$$

the point A does not lie on the hyperplane h:

$$\boxed{A \notin h}$$

white vertex

$\circ$

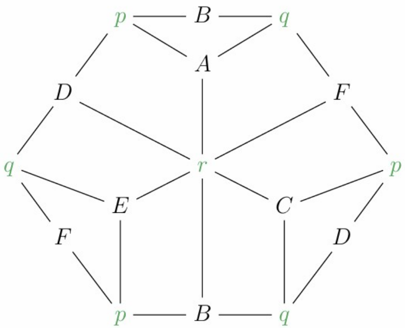
$h \in \mathbb{P}^*$

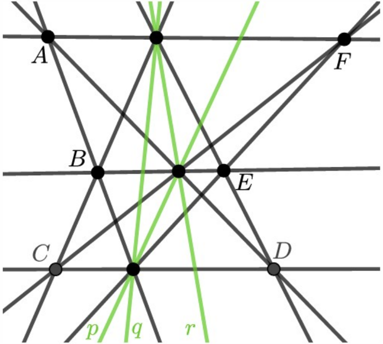
associated with a **hyperplane**

Master Theorem

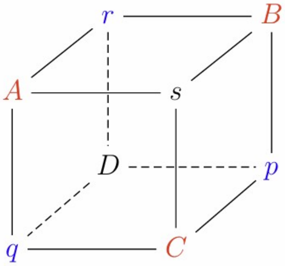
If all tiles of T, with the exception of one, are coherent, then the remaining tile is coherent as well.

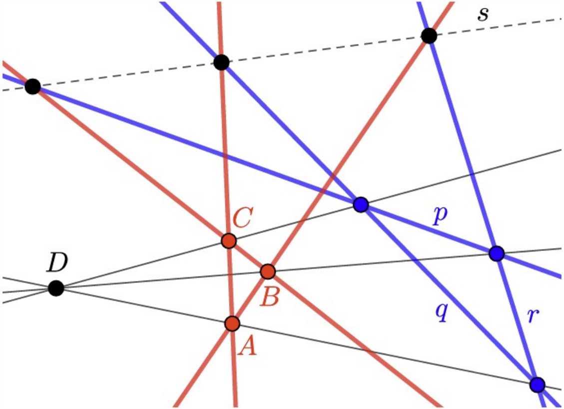
Master Proof of Pappus's Theorem



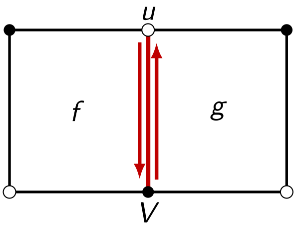


Master Proof of Desargues's Theorem





Proof of the Master Theorem



$f : u \to V, \qquad g : V \to u$

same edge uV, opposite orientations

For the shared edge uV,

$$< u, V >< u, V >^{-1} = 1.$$

$$\boxed{\prod_{\text{faces } f} \chi(f) = 1}$$

$$\chi(f) = 1 \iff f \text{ coherent.}$$

$$\boxed{\chi(f) = 1 \ (f \neq f_0) \implies \chi(f_0) = 1}$$

Attention:

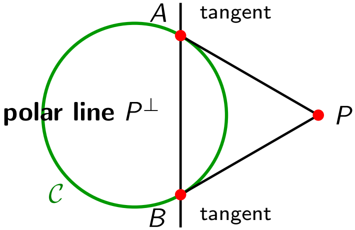
There is a 2-coloring of the Surface

and the surface must be **oriented**.(Question: How about nonoriented case?)

Projective Polarity

$$\boxed{\beta(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2 - x_3y_3, \qquad \mathcal{C} : \beta(\mathbf{x},\mathbf{x}) = 0}$$

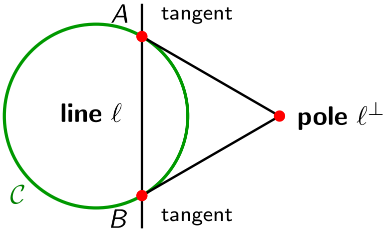
Point $\longrightarrow$ Polar Line



$$P \longrightarrow A, B \longrightarrow \boxed{P^\perp = AB}$$

$$\boxed{\beta(A,P) = \beta(B,P) = 0}$$

Line $\longrightarrow$ Pole

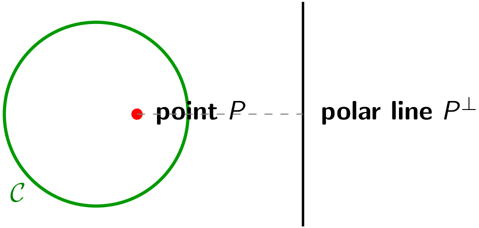


$$\ell \cap \mathcal{C} = \{A, B\} \longrightarrow \boxed{\ell^\perp = PA \cap PB}$$

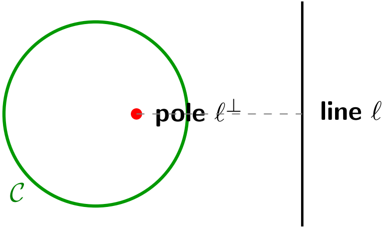
$$\boxed{\beta(A,P) = \beta(B,P) = 0}$$

Projective Polarity

Interior Point $\longrightarrow$ **Exterior Polar Line**



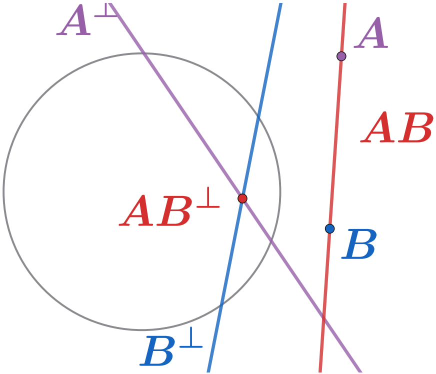
Exterior Line $\longrightarrow$ **Interior Pole**



$$P \in \mathrm{int}(\mathcal{C}) \quad \Longrightarrow \quad \boxed{P^\perp \cap \mathcal{C} = \varnothing}$$

$$\ell \cap \mathcal{C} = \varnothing \quad \Longrightarrow \quad \boxed{\ell^\perp \in \mathrm{int}(\mathcal{C})}$$

Projective Polarity



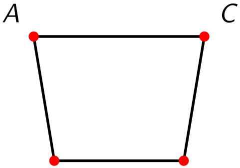
$$A, B \in (AB)$$

$$\Updownarrow$$

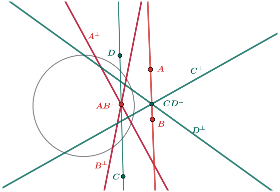
$$\boxed{(AB)^\perp = A^\perp \cap B^\perp}$$

Coherent Polarity Tile and Mixed Cross Ratio of Polarity

MCRP: $\chi_\beta(A, B; C, D) = \dfrac{\beta(A, C)\beta(B, D)}{\beta(B, C)\beta(A, D)}$



D **polarity tile** B
$(A, B; C, D)$



$A,\ B,\ (CD)^\perp$ are collinear.

$$\underbrace{\chi_\beta(A, B; C, D) = 1}_{\text{coherent}}$$

$$\Longleftrightarrow$$

$$\boxed{AB \perp_\beta CD}$$

$$\Longleftrightarrow$$

$A,\ B,\ (CD)^\perp$ are collinear

$$\Longleftrightarrow$$

$C,\ D,\ (AB)^\perp$ are collinear

$$\Longleftrightarrow$$

$A^\perp,\ B^\perp,\ CD$ are concurrent

The Master Theorem of Polarity [Jiang–L.–L.26+]

Polarity Master Theorem.
Consider a labeled quadrilateral tiling Q of a connected closed surface **(Oriented or Nonoriented)**.

Assume

vertices labeled by projective points, adjacent labels not β-conjugate,

$$\text{closed cycle } \gamma : \quad \begin{cases} \text{same orientation on return} & \Longrightarrow |\gamma| \text{ even}, \\ \text{opposite orientation on return} & \Longrightarrow |\gamma| \text{ odd}. \end{cases}$$

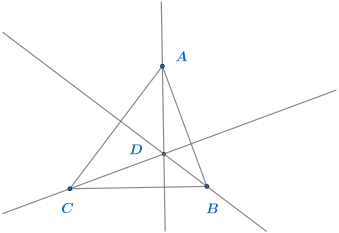
Then

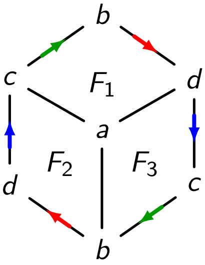
$$\boxed{\prod_{\text{faces } f} \chi_\beta(f) = 1}$$

Consequently

$$f \text{ coherent} \quad \Longleftrightarrow \quad \chi_\beta(f) = 1,$$

$$\boxed{\text{all but one face coherent} \quad \Longrightarrow \quad \text{remaining face coherent}}$$





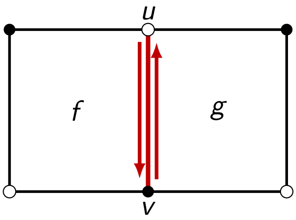
Hesse tiling on $\mathbb{RP}^2$

$$
\begin{aligned}
F_1 &: \quad ab \perp_\beta cd, \\
F_2 &: \quad ad \perp_\beta bc, \\
F_3 &: \quad ac \perp_\beta bd.
\end{aligned}
$$

Hesse Theorem

$$
\left.\begin{array}{l} F_1 \text{ coherent} \\ F_3 \text{ coherent} \end{array}\right\} \Longrightarrow F_2 \text{ coherent.}
$$

Proof of the Master Theorem of Polarity



$f : u \to v, \qquad g : v \to u$

same edge uv, opposite orientations

Oriented Surface:
Same as Fomin, Pylyavskyy's

Nonoriented Surface:

$$S \xrightarrow{\textbf{Orientation double cover}} \tilde{S}$$

do cancellation to $\tilde{S}$

Coherent Polygons [Fomin–Pylyavskyy, 2023]

Consider a topological disk **P** with $2n$ marked points on its boundary, labeled cyclically by

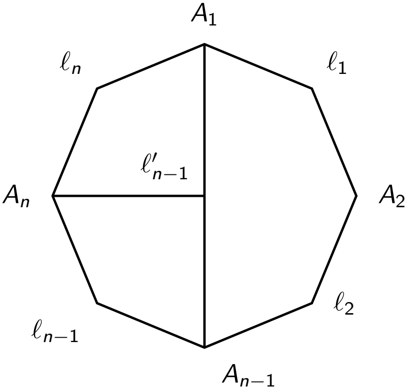
$$A_1, \ell_1, A_2, \ell_2, \ldots, A_n, \ell_n.$$

We call **P coherent** if

$$(A_1, \ldots, A_n; \ell_1, \ldots, \ell_n) = 1,$$

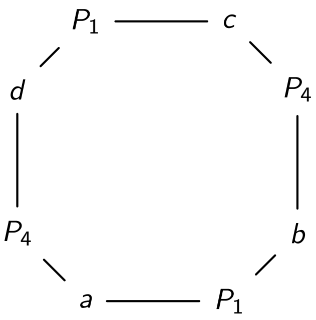
where

$$(A_1, \ldots, A_n; \ell_1, \ldots, \ell_n) = \frac{\langle \mathbf{A}_1, \boldsymbol{\ell}_1 \rangle \langle \mathbf{A}_2, \boldsymbol{\ell}_2 \rangle \cdots \langle \mathbf{A}_n, \boldsymbol{\ell}_n \rangle}{\langle \mathbf{A}_2, \boldsymbol{\ell}_1 \rangle \langle \mathbf{A}_3, \boldsymbol{\ell}_2 \rangle \cdots \langle \mathbf{A}_1, \boldsymbol{\ell}_n \rangle}.$$

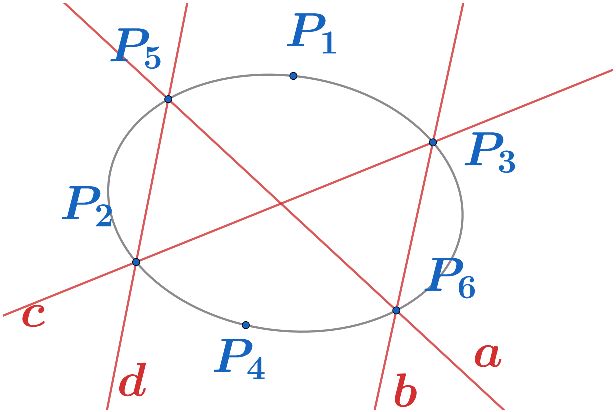


alternating labels A_i, ℓ_i around the boundary

Example 2: Coherent Polygons of a Conic



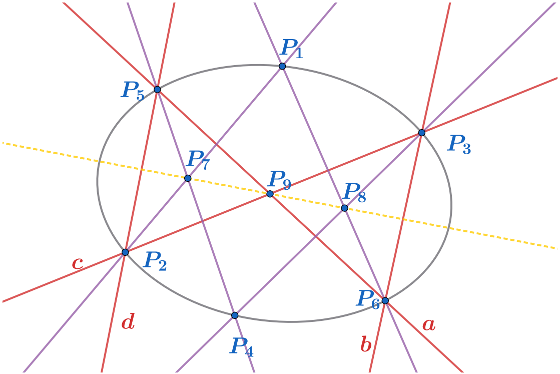
A coherent $2n$-gon, $n = 4$.



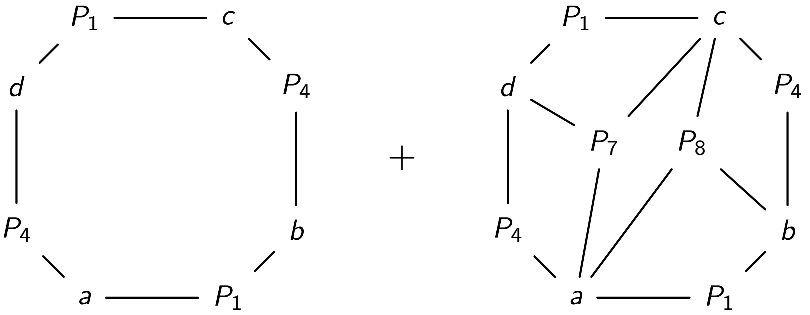
Example 3: Pascal Theorem

Pascal Theorem. Let $P_1, P_2, \ldots, P_6$ be six points on a conic. Then

$$P_7 = P_1P_2 \cap P_4P_5 \qquad P_8 = P_2P_3 \cap P_5P_6 \qquad P_9 = P_3P_4 \cap P_6P_1 \qquad \Rightarrow P_7, P_8, P_9 \text{ collinear.}$$



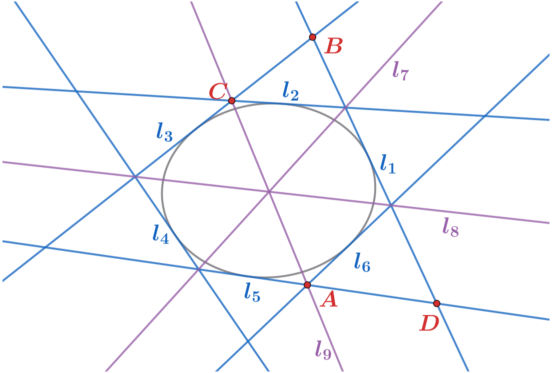
Proof of Pascal Theorem



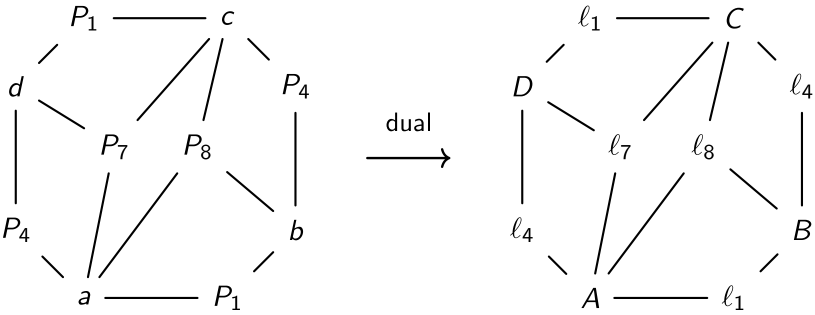
Example 4: Brianchon Theorem

Brianchon Theorem. Let $\ell_1, \ell_2, \ldots, \ell_6$ be six tangents on a conic. Then

$\ell_7 = (\ell_1 \cap \ell_2)\cdot(\ell_4 \cap \ell_5)$ $\quad \ell_8 = (\ell_2 \cap \ell_3)\cdot(\ell_5 \cap \ell_6)$ $\quad \ell_9 = (\ell_3 \cap \ell_4)\cdot(\ell_6 \cap \ell_1)$ $\quad \Rightarrow \ell_7, \ell_8, \ell_9$ concurrent.



Proof of Brianchon Theorem



The twist restores global cancellation

Orientation-twisted

Every even cycle is orientation-preserving.
Every odd cycle is orientation-reversing.

Polarity Master Theorem

$$\prod_{f \in F(Q)} \chi_\beta(f) = 1.$$

Hence coherence of all but one face forces coherence of the last face.

A Klein Bottle Tiling

For $A_1, A_2, A_3, B_1, B_2, B_3$, define

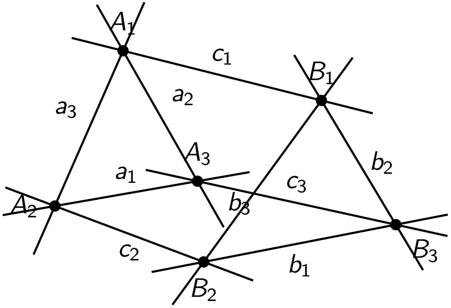
$$a_i = A_j A_k, \qquad b_i = B_j B_k, \qquad c_i = A_i B_i, \qquad \{i, j, k\} = \{1, 2, 3\}.$$

The six conditions are

$$a_i \perp_\beta c_i, \qquad b_i \perp_\beta c_i \qquad (i = 1, 2, 3).$$

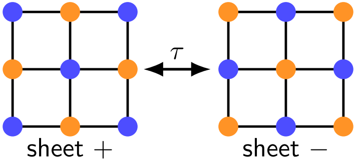
Any five of these conditions imply the sixth one.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, dash] & A_3 \arrow[r] \arrow[d, dash] & B_2 \arrow[d] \\
B_1 \arrow[r, dash] \arrow[d] & B_3 \arrow[r, dash] \arrow[d, dash] & A_2 \arrow[r, dash] \arrow[d, dash] & A_1 \arrow[d] \\
A_3 \arrow[r] & B_2 \arrow[r] & B_1 \arrow[r] & A_3
\end{tikzcd}$$



Orientation Double Cover

Q: Non-orentable tiling, $\tilde{Q}$: orientation double cover, τ: sheet involution



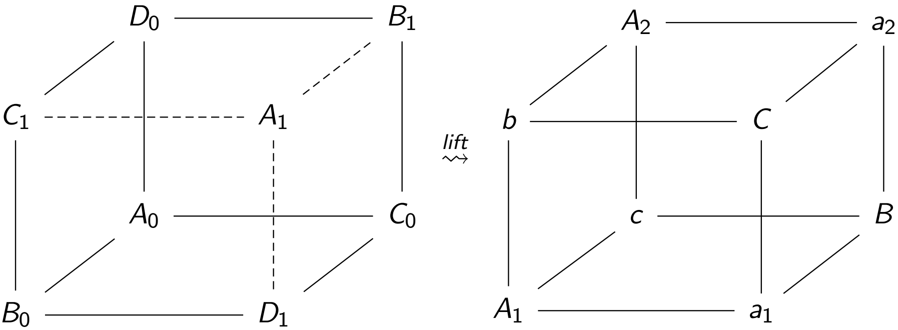
$\forall$ two-coloring on the lifted tiling, τ switchs colors.

A polarity incidence theorem lifts to a self-dual incidence theorem

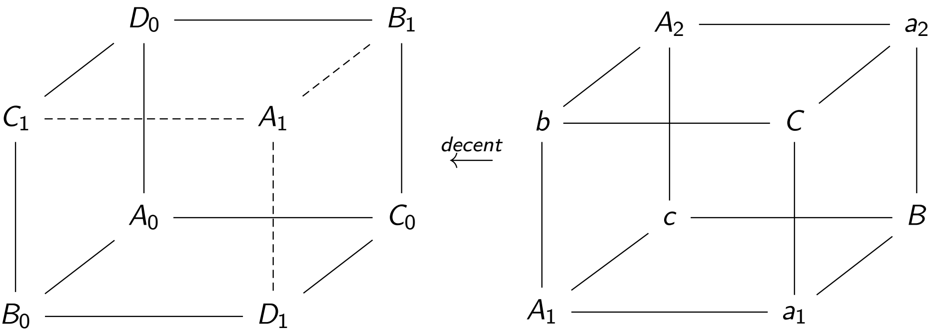
$$\boxed{\bigwedge_{g\in F(Q)\setminus\{f\}} I(g) \Longrightarrow I(f)} \overset{\textit{lift}}{\rightsquigarrow} \boxed{\bigwedge_{g\in F(\widetilde{Q})\setminus\{\widetilde{f},\tau\widetilde{f}\}} I(g) \Longrightarrow \left(I(\widetilde{f}) \Longleftrightarrow I(\tau\widetilde{f})\right)}$$

Hesse: hemicube on $\mathbb{RP}^2 \longleftarrow$ Desargues: cube on S^2

Möbius band $\longleftarrow$ annulus.



Descent is controlled by an admissible involution



- fixed-point-free and orientation-reversing;
- color-swapping: $c \circ \tau = c + 1$;
- such that its orbit cell structure is a regular quadrangulation.

Topology and Theorem Based Ring

Theorem (Non-commutative Master Theorem, Pylyavskyy–Skopenkov 26, Izosimov 26)

If a incidence theorem has a sphere surface-tiling proof, then it is true over any skew field.

Proposition (Fomin–Pylyavskyy 23)

Pappus' theorem has a torus-tiling proof but is false over any skew field.

Topology Classifies Excision

Theorem (Jiang–L.–L.26+)

For divion ring D, σ-hermitian h, tiling surfaces restrict generalization of D, σ:

- S^2: *arbitrary* (D, σ);
- Σ_g, $g \geq 1$: D *commutative, arbitrary* σ;
- N_k, $k \geq 1$: D *commutative and* $\sigma = id$.

Hermitian Hesse analogue fails

Over $(\mathbb{C}, {}^{-})$, one Hermitian configuration has

$$
\begin{aligned}
A &= (1, 0, 0), \\
B &= (1, 1, 0), \\
C &= (1, i, 1), \\
D &= (1, i, -1 - i).
\end{aligned}
$$

$AB \perp_h CD, AC \perp_h BD$ but $AD \not\perp_h BC$

Fundamental Group and Euler Genus

Define

$$sl^{odd}_{G_I}(\delta_I) = \min\{k : \delta_I = z_1^2 \cdots z_k^2,\ z_i \text{ are odd}\},$$

$$cl^{even}_{G_I}(\delta_I) = \min\{g : \delta_I = \prod_{i=1}^{g}[a_i, b_i],\ a_i, b_i \text{ are even}\}.$$

Theorem (Euler genus formula, Jiang–L.–L.26+)

$$\boxed{eg_{el}(I) = tl_{G_I}(\delta_I) = \min\left\{sl^{odd}_{G_I}(\delta_I), 2cl^{even}_{G_I}(\delta_I)\right\}.}$$

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